

A geometric Hamilton–Jacobi theory on a Nambu–Jacobi manifold

M. de León and C. Sardón

Instituto de Ciencias Matemáticas, Campus Cantoblanco

Consejo Superior de Investigaciones Científicas

C/ Nicolás Cabrera, 13–15, 28049, Madrid. SPAIN

Abstract

In this paper we propose a geometric Hamilton–Jacobi theory on a Nambu–Jacobi manifold. The advantage of a geometric Hamilton–Jacobi theory is that if a Hamiltonian vector field X_H can be projected into a configuration manifold by means of a one-form dW , then the integral curves of the projected vector field X_H^{dW} can be transformed into integral curves of the vector field X_H provided that W is a solution of the Hamilton–Jacobi equation. This procedure allows us to reduce the dynamics to a lower dimensional manifold in which we integrate the motion. On the other hand, the interest of a Nambu–Jacobi structure resides in its role in the description of dynamics in terms of several Hamiltonian functions. It appears in fluid dynamics, for instance. Here, we derive an explicit expression for a geometric Hamilton–Jacobi equation on a Nambu–Jacobi manifold and apply it to the third-order Riccati differential equation as an example.

1 Motivation

The Hamilton–Jacobi equation (HJ equation) constitutes the third complete formulation of classical mechanics, after Newtonian and Hamiltonian mechanics. It is a first-order partial differential equation for a generating function $S(q^i, t)$ on a n -dimensional configuration manifold Q with local canonical coordinates $\{q^i, i = 1, \dots, n\}$ and $H = H(q^i, p_i)$ is the Hamiltonian function of the system on T^*Q , which is locally coordinated by $\{q^i, p_i, i = 1, \dots, n\}$. Explicitly,

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0. \quad (1)$$

This equation is particularly useful for the identification of conserved quantities and roots in the philosophy of finding a canonical transformation

with generating function $S(q^i, t)$ that leads to the equilibrium of a mechanical system [1, 2]. The generating function $S(q^i, t)$ is also interpreted as the action of a variational principle,

$$S = \int_{(q_1, t_1)}^{(q_n, t_n)} L(q(t), \dot{q}(t), t) dt \quad (2)$$

such that the condition $\frac{\delta S}{\delta t} = 0$ retrieves the Hamiltonian equations [11].

It is possible to separate the temporal dependency on S through the Ansatz $S = W(q^1, \dots, q^n) - Et$, where E is the total energy of the system. This choice gives rise to the time-independent HJ equation [1, 11], which can be interpreted geometrically:

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E. \quad (3)$$

Concerning the geometric interpretation of a HJ theory, the primordial observation is, on a symplectic phase space, that if a Hamiltonian vector field $X_H : T^*Q \rightarrow TT^*Q$ can be projected into a vector field $X_H^{dW} : Q \rightarrow TQ$ on a lower dimensional manifold by means of a 1-form dW , then the integral curves of the projected vector field X_H^{dW} can be transformed into integral curves of X_H provided that W is a solution of (3). If we define the projected vector field as:

$$X_H^{dW} = T\pi \circ X_H \circ dW, \quad (4)$$

where $T\pi$ is the induced projection on the tangent space, $T\pi : TT^*Q \rightarrow T^*Q$ by the canonical projection $\pi : T^*Q \rightarrow Q$, it implies the commutativity of the diagram below:

$$\begin{array}{ccc} T^*Q & \xrightarrow{X_H} & TT^*Q \\ \downarrow \pi & & \downarrow T\pi \\ Q & \xrightarrow{X_H^{dW}} & TQ \end{array}$$

*(Note: A curved arrow labeled dW points from Q to T^*Q .)*

provided that dW is a Lagrangian submanifold, since dW is exact and then it is closed. This condition gave rise to the introduction of Lagrangian submanifolds in dynamics was necessary. Lagrangian submanifolds are very important objects in Hamiltonian mechanics, since the dynamical equation (hamiltonian or lagrangian) can be described as lagrangian submanifolds of convenient symplectic manifolds.

The pioneer in this purpose was Tulczyjew who characterized the image of local Hamiltonian vector fields on a symplectic manifold (M, ω) as lagrangian submanifolds of a symplectic manifold (TM, ω^T) , where ω is the tangent lift of ω to TM [34]. This result was later generalized to Poisson manifolds [13] and Jacobi manifolds [15].

Using the approach discussed above, the HJ theory has also been extended to nonholonomic mechanics, geometric mechanics on Lie algebroids, singular systems, control theory, classical field theories and different geometric backgrounds [5, 6, 22, 18, 19, 21]. This proves the wide applicability of the geometric interpretation of the HJ theory and its recent interest among the scientific community.

In this paper, we deal with Nambu–Jacobi manifolds (NJ manifolds). The NJ structure is a generalization of Nambu–Poisson structures (NP structures) and both appeared as an extension of mechanics on even and odd higher-dimensional phase spaces. In particular, both NJ and NP structures include n -dimensional brackets which are very useful for descriptions of physical systems equipped with several Hamiltonian functions. These structures have also played an important role in the study of Dirac’s constraints and Nambu’s mechanics [3]. The first bracket of order $n > 2$ was the original three-dimensional Nambu bracket [28], defined in terms of a Jacobian:

$$\{H_1, H_2, f\} = \frac{\partial(H_1, H_2, f)}{\partial(x, y, z)}, \quad (5)$$

with canonical variables satisfying $[x, y, z] = 1$ and interpreted as a bracket defined by the canonical volume form in \mathbb{R}^3 . This bracket served as a bracket of a three-dimensional phase space for the dynamics of particles composed by three quarks [28]. Also, it has showed its applicability in noncanonical theories of perfect fluid dynamics [14, 29]. For example, consider Eulerian variables for an incompressible fluid in 3D governed by the vorticity equation:

$$\frac{\partial \Omega}{\partial t} + (u \nabla) \Omega - (\Omega \nabla) u = 0, \quad (6)$$

where $u = (u_1, u_2, u_3)$ is the velocity, the vorticity is $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and they are related by $\Omega = \nabla \times u$ and $\nabla u = 0$. The total energy

$$H = \frac{1}{2} \int \Omega u d^3x = -\frac{1}{2} \int \Omega A d^3x \quad (7)$$

and the helicity

$$h = \frac{1}{2} \int \Omega u d^3x \quad (8)$$

are conserved, assuming that u vanishes at infinity and A is a vector potential such that $u = -\nabla \times A$, $\nabla A = 0$. The energy and helicity are Casimir functions acting like Hamiltonians for Nambu mechanics. Now, the evolution of an arbitrary functional $F = F(\Omega)$ will be given by the Nambu bracket defined in (5):

$$\frac{dF}{dt} + \{F, h, H\} = 0. \quad (9)$$

Afterwards, a generalization of the three-dimensional Nambu bracket to an n -order bracket was provided [28, 33], as the n -dimensional Jacobian:

$$\{H_1, \dots, H_{n-1}, f\} = \frac{\partial(H_1, \dots, H_{n-1}, f)}{\partial(x^1, \dots, x^n)}. \quad (10)$$

Many hierarchies of differential equations are endowed with a recursion relation that generates subsequent members of the hierarchy, conserved quantities and multiple compatible Hamiltonian functions describing the differential equation. For example, in [9] it is shown that the dispersionless Toda hierarchy can be described in terms of multiple compatible Hamiltonian functions,

$$H_n = \int h_n(u) dx, \quad h_n = (n+1)^{-1} Q_{n+1}, \quad (11)$$

where Q are symmetric polynomials of u and depend on a zero-curvature metric. The evolution of an observable, let us say f , is governed by the n -dimensional bracket (10) of the observable and Hamiltonian functions in (11).

Later, the Nambu bracket was generalized by Takhtajan [33] by imposing the fundamental identity:

$$\begin{aligned} \{H_1, \dots, H_{n-1}, \{g_1, \dots, g_n\}\} = \\ \sum_{i=1}^n \{g_1, \dots, g_{i-1}, \{H_1, \dots, H_{n-1}, g_i\}, g_{i+1}, \dots, g_n\} \end{aligned} \quad (12)$$

for real valued and infinitely differentiable functions $H_1, \dots, H_{n-1}, g_1, \dots, g_n$. Expression (12) is merely an extension of the well-known Jacobi identity. This led him to the introduction of a NP manifold as a manifold whose ring of functions is endowed with an n -dimensional bracket (10) that satisfies (12). This fundamental identity was independently considered by other authors (see [31]). Moreover, these axioms are just the ones of an n -Lie algebra

introduced Filippov [10] in 1985. The extension of this bracket to an analogous of Jacobi bracket was simultaneously considered by several authors [12, 16, 17, 27].

The HJ theory for NP manifolds was devised by the present authors in [26]. Our aim here is to proceed similarly in the case of a NJ manifold. In this way, the outline of the paper is the following: in Section 2, we recall the fundamentals on the geometry of NJ manifolds, their structural theorem and present dynamics on such manifolds. In particular, we focus on a particular case of NJ manifolds, that is the case of volume NJ manifolds. For the development of a geometric HJ theory, we introduce the notion of Lagrangian submanifolds of a NJ manifold. In Section 3, we develop a geometric HJ theory on volume NP manifolds and obtain an explicit expression for a HJ equation. To finish, in Section 4, we propose an application of the geometric HJ theory on volume NJ manifolds through an example of physical interest: a four-dimensional system of Riccati first-order differential equations. We derive a volume NJ structure for such equation and then apply the HJ theory. A particular solution for the equation is indicated.

We introduce the following notation: we will denote the algebra of C^∞ real-valued functions as $C^\infty(M, \mathbb{R})$, $\mathfrak{X}(M)$ is the $C^\infty(M, \mathbb{R})$ module of vector fields on M . The space $\mathcal{V}^k(M)$ is the space of k -vectors on M and $\Omega^k(M)$ is the space of k -forms on M .

Assume all mathematical objects to be C^∞ , globally defined and that manifolds are connected. This permits us to suit technical details while highlighting the main aspects of the theory.

2 Nambu–Jacobi manifolds

Let us consider a generalized almost Jacobi manifold $(M, \{\dots\})$ where M is an m -dimensional manifold and the bracket is a generalized almost Jacobi bracket $\{\dots\} : C^\infty(M, \mathbb{R}) \times \dots^n \times C^\infty(M, \mathbb{R})$ of order n . It has two fundamental properties:

$$\{f_1, \dots, f_n\} = \epsilon_\sigma \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\},$$

that implies that the bracket is skew-symmetric and it is a first-order linear differential operator on M with respect to each argument

$$\begin{aligned} \{f_1 g_1, f_2 \dots, f_n\} &= f_1 \{g_1, f_2, \dots, f_n\} \\ &\quad + g_1 \{f_1, f_2, \dots, f_n\} - f_1 g_1 \{1, f_2, \dots, f_n\} \end{aligned}$$

for f_1, \dots, f_n and $g_1 \in C^\infty(M, \mathbb{R})$. We can define the multivectors $\Lambda \in \mathcal{V}^n(M)$ and $\square \in \mathcal{V}^{n-1}(M)$ as follows:

$$\begin{aligned}\square(df_1, \dots, df_{n-1}) &= \{1, f_1, \dots, f_{n-1}\}, \\ \Lambda(df_1, \dots, df_n) &= \{f_1, \dots, f_n\} + \sum_{i=1}^n (-1)^i f_i \square(df_1, \dots, d\hat{f}_i, \dots, df_n)\end{aligned}$$

Conversely, any pair $(\Lambda, \square) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ defines a generalized almost Jacobi bracket of order n on M , given by:

$$\begin{aligned}\{f_1, \dots, f_n\} &= \Lambda(df_1, \dots, df_n) \\ &\quad + \sum_{i=1}^n (-1)^{i-1} f_i \square(df_1, \dots, d\hat{f}_i, \dots, df_n)\end{aligned} \quad (13)$$

such that $(M, \{\dots\})$ a generalized almost Jacobi structure. If $\square = 0$, we recover a generalized almost Poisson structure. Furthermore, if we add the integrability condition

$$\begin{aligned}\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} \\ = \sum_{i=1}^n \{g_1, \dots, g_{i-1}, \{f_1, \dots, f_{n-1}, g_i\}, g_{i+1}, \dots, g_n\},\end{aligned} \quad (14)$$

we have that the pair $(M, \{\dots\})$ is a Nambu–Jacobi structure, or equivalently, we say that (M, Λ, \square) is a Nambu–Jacobi manifold. In the case that $\square = 0$, we have a Nambu–Poisson structure.

We define a vector field $X_{H_1, \dots, H_{n-1}}^\Lambda$, understood as the vector field on a NP manifold (M, Λ) as:

$$X_{f_1, \dots, f_{n-1}}^\Lambda = \sharp_\Lambda(df_1 \wedge \dots \wedge df_{n-1}) \quad (15)$$

and define the morphism $\sharp_\Lambda : \Lambda^{n-1}(M) \rightarrow \mathfrak{X}(M)$ induced by Λ . Similarly, $X_{f_1, \dots, f_{n-1}}^\square$ can be understood as a vector field on a NP manifold (M, \square) as:

$$X_{f_1, \dots, \hat{f}_i, \dots, f_{n-1}}^\square = \sharp_\square(df_1 \wedge \dots \wedge d\hat{f}_i \wedge \dots \wedge df_{n-1}) \quad (16)$$

and defines the morphism $\sharp_\square : \Lambda^{n-2}(M) \rightarrow \mathfrak{X}(M)$ induced by \square . The operators (Λ, \square) induce the following pairings, correspondingly:

$$\Lambda(\alpha_1, \dots, \alpha_{n-1}, \beta) = \langle \sharp_\Lambda(\alpha_1, \dots, \alpha_{n-1}), \beta \rangle \quad (17)$$

and

$$\square(\alpha_1, \dots, \alpha_{n-2}, \beta) = \langle \sharp_\square(\alpha_1, \dots, \alpha_{n-2}), \beta \rangle \quad (18)$$

for $\alpha_1, \dots, \alpha_{n-1}, \beta \in \Omega^1(M)$. A vector field on a NJ manifold (M, Λ, \square) is defined as:

$$X_{f_1, \dots, f_{n-1}} = X_{f_1, \dots, f_{n-1}}^\Lambda + \sum_{i=1}^{n-1} (-1)^{i-1} f_i X_{H_1, \dots, \widehat{f_i}, \dots, f_{n-1}}^\square. \quad (19)$$

So, we define the characteristic distribution on a point $x \in M$ for a NJ manifold (M, Λ, \square) as:

$$\mathcal{C}_x = \langle X_{f_1, \dots, f_{n-1}}(x) \rangle = \text{Im}(\sharp_\Lambda((x))) + \text{Im}(\sharp_\square(x)). \quad (20)$$

Now, let us point out some properties of NJ manifolds.

Properties:

Let (M, Λ, \square) be a NJ manifold, M has dimension m , Λ is of order n and \square of order $(n-1)$ and such that $n > 2$. Then,

1. (M, Λ) is a NP structure on M of order n .
2. (M, \square) is a NP structure on M of order $n-1$.
3. For every point $x \in M$ where $\Lambda(x) \neq 0$, there exists a one-form $\theta_x \in T^*M$ such that

$$\iota_{\theta_x} \Lambda(x) = \square(x). \quad (21)$$

4. The Lie derivative $\mathcal{L}_{X_{f_1, \dots, f_{n-2}}^\square} \Lambda = 0$.

2.1 Volume Nambu–Jacobi manifolds

A volume Nambu–Jacobi manifold (M, Ω, θ) , (henceforth VNJ manifold) is a Nambu–Jacobi manifold in which $\dim(M) = n$, where Ω is a volume form on M and θ is a one-form that is closed, $d\theta = 0$. There is an associated pair $(\Lambda_\Omega, \square_\Omega) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ for a VNJ manifold, operating as follows:

$$\Lambda_\Omega(df_1, \dots, df_n) = \{f_1, \dots, f_n\}_{\Lambda_\Omega} \quad (22)$$

and the $(n-1)$ -order skew symmetric tensor \square_Ω defined by:

$$\square_\Omega = \iota_\theta \Lambda_\Omega \quad (23)$$

where the bracket induced by Λ_Ω is defined by:

$$\Omega\{f_1, \dots, f_n\}_{\Lambda_\Omega} = df_1 \wedge \dots \wedge df_n \quad (24)$$

A particular example is $M = \mathbb{R}^n$. Choosing canonical coordinates $\{x^i, i = 1, \dots, n\}$, the volume form here is $\Omega_{\mathbb{R}^n} = dx^1 \wedge \dots \wedge dx^n$, the multivectors $(\Lambda, \square) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ take the form:

$$\Lambda_{\mathbb{R}^n} = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}, \quad \square_{\mathbb{R}^n} = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^{n-1}}. \quad (25)$$

and θ is locally expressed as $\theta_{\mathbb{R}^n} = (-1)^n dx^n$. The brackets induced by $(\Lambda_\Omega, \square_\Omega)$ on a volume manifold (M, Ω, θ) take the form:

$$\{f_1, \dots, f_n\}_{\Lambda_{\mathbb{R}^n}} = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^n)}, \quad (26)$$

and

$$\{f_1, \dots, \widehat{f}_i, \dots, f_n\}_{\square_{\mathbb{R}^n}} = \frac{\partial(f_1, \dots, \widehat{f}_i, \dots, f_n)}{\partial(x^1, \dots, x^{n-1})}. \quad (27)$$

Definition 1. Consider a VNJ manifold (M, Ω, θ) and $H_1, \dots, H_{n-1} \in C^\infty(M, \mathbb{R})$. Then, the expression in coordinates on \mathbb{R}^n for a Hamiltonian vector field on a NJ manifold (19) is:

$$\begin{aligned} X_{H_1, \dots, H_{n-1}} &= \sum_{k=1}^{n-1} (-1)^{n-k} \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, \widehat{x}^k, \dots, x^n)} \frac{\partial}{\partial x^k} + \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-1})} \frac{\partial}{\partial x^n} \\ &+ \sum_{i=1}^{n-1} (-1)^{i-1} H_i \frac{\partial(H_1, \dots, \widehat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-2})} \frac{\partial}{\partial x^{n-1}} \\ &+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} (-1)^{n-j+i-2} H_i \frac{\partial(H_1, \dots, \widehat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, \widehat{x}^j, \dots, x^{n-1})} \frac{\partial}{\partial x^j}. \end{aligned} \quad (28)$$

This is a Hamiltonian system on a VNJ manifold.

We can extend the model of VNJ manifolds proposed to a more general situation. If (M, Λ) is a NP with $n > 2$ and θ is a closed one-form on M , then $(M, \Lambda, \iota_\theta \Lambda)$ is a NJ structure of order n on M . Furthermore, if Λ_Ω is a tensor of order $n > 2$ associated with a volume form Ω on M and θ is a one-form, then the triple $(M, \Lambda_\Omega, \iota_\theta \Lambda_\Omega)$ is a NJ structure [17]. Now, let us now introduce the structural theorem of NJ manifolds [17]. The structure in leaves of the foliation encompasses the former examples, which are canonical examples for these manifolds.

Theorem 2 (Structure theorem). *Let (M, Λ, \square) be a Nambu–Jacobi manifold of order $n \geq 3$ and $x \in M$. Suppose that \mathcal{D} is a characteristic foliation of M and that L is the leaf of \mathcal{D} passing through x . Then, (Λ, \square) reduces to a Nambu–Jacobi structure (Λ_L, \square_L) on L ,*

1. *If $\Lambda(x) \neq 0$, then L has dimension n associated with a volume form on L and there exists a closed one-form θ_L on L such that $\square_L = \iota_{\theta_L} \Lambda_L$.*
2. *If $\Lambda(x) = 0$ and $\square(x) \neq 0$, then L has dimension $(n - 1)$ and $\Lambda_L = 0$. Moreover,*
 - *If $n > 3$, then \square_L is a Nambu–Poisson structure of order $(n - 1)$ associated with a volume form on L .*
 - *If $n = 3$, then \square_L is a symplectic structure.*
3. *If $\Lambda(x) = 0$ and $\square(x) = 0$, then $L = \{0\}$ and the induced Nambu–Jacobi structure is trivial.*

Proof. Proof of this theorem can be found in reference [17], where it was formerly stated. \square

2.2 Lagrangian submanifolds

Let (M, Λ, \square) be a NJ manifold. We say that a submanifold $N \subset M$ is j -lagrangian $\forall x \in N$, $1 \leq j \leq n - 1$, if:

$$\sharp_{\Lambda} \text{Ann}^j(T_x N) = \mathcal{C}_x \cap T_x N \quad (29)$$

where the annihilator is defined as:

$$\text{Ann}^j(T_x N) = \{\alpha \in \Lambda^{n-1}(T_x^* M) \mid \iota_{v_1 \wedge \dots \wedge v_j} \alpha = 0, \forall v_1, \dots, v_j \in T_x N\} \quad (30)$$

and recall the definition of \mathcal{C}_x in (20).

In the particular case of a VNJ manifold, expression (29) reduces to:

$$\sharp_{\Lambda} \text{Ann}^j(T_x N) = \sharp_{\Lambda} \Lambda^{n-1}(T_x^* M) \cap T_x N. \quad (31)$$

Theorem 3. *Given a VNJ structure (M, Ω, θ) , every submanifold N of dimension $(n - 1)$ is $(n - 1)$ -lagrangian. No other lagrangian submanifolds exist.*

Proof. Recall the expressions in coordinates (25) for the pair (Λ, \square) . Considering the definition in (29), we compute the term $\sharp_\Lambda \Lambda^{n-1}(T_x^* M)$. An element α of $\Lambda^{n-1}(T_x^* M)$ has the local expression

$$\alpha = \sum_{i=1}^n \alpha_{1\ldots\hat{i}\ldots n} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n \quad (32)$$

where $\widehat{dx^i}$ stands for the omitted term dx^i and $\alpha_{1\ldots\hat{i}\ldots n}$ is the coefficient of the $(n-1)$ -order form $\alpha \in \Lambda^{n-1}(T_x^* M)$ associated with the combination $dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n$. Then,

$$\text{Im}(\alpha) = \sum_{i=1}^n (-1)^{n-i} \alpha_{1\ldots\hat{i}\ldots n} \frac{\partial}{\partial x^i} \quad (33)$$

and therefore,

$$\sharp_\Lambda \Lambda^{n-1}(T_x^* M) = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\rangle.$$

This implies that $\sharp_\Lambda \Lambda^{n-1}(T_x^* M) = T_x M = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\rangle$ and hence, the right-hand side of expression (29) in this particular case equals

$$\sharp_\Lambda \Lambda^{n-1}(T_x^* M) \cap T_x N = T_x N$$

Computing the left-hand side term of (29), let us assume that N is $(n-1)$ -dimensional and the equations of the submanifold are $\{x^n = 0\}$ in the coordinates we gave chosen. So, the tangent space to this submanifold is generated by $TN = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}} \right\rangle$. The elements of the annihilator when $j = n-1$ are $\alpha \in \Lambda^{n-1}(T_x^* M)$ such that

$$\iota_{\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}}} \alpha = 0, \quad \forall \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}} \in T_x N \quad (34)$$

The α 's fulfilling this condition are of the form

$$\text{Ann}^{n-1}(TN) = \sum_{i=1}^{n-1} \alpha_{1\ldots\hat{i}\ldots n} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n. \quad (35)$$

Hence, the annihilator of order $j-1$ for a submanifold N of dimension $n-1$ is generated by

$$\text{Ann}^{n-1}(TN) = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}} \right\rangle = TN \quad (36)$$

This means that the left-hand side and the right-hand side of (29) are equal, and the submanifold $N \subset M$ of dimension $n-1$ is a Lagrangian submanifold of order $n-1$. It is easy to see that $\text{Ann}^{j \neq n-1}(T_x N) = \{0\}$ and that for submanifolds N of dimension different from $n-1$ the annihilator of any order j equals zero, namely, $\text{Ann}^j(T_x N) = \{0\}$, for $\dim(M) \neq n-1$. \square

3 Hamilton-Jacobi theory on VNJ manifolds

Consider a VNJ structure (M, Ω, θ) , a fibration $\pi : M \rightarrow N$ such that $\dim(M) = n$ and $\dim(N) = n-1$ and a section $\gamma : N \rightarrow M$. The projected vector field on N can be defined as:

$$X_{H_1, \dots, H_{n-1}}^\gamma = T_\pi \circ X_{H_1, \dots, H_{n-1}} \circ \gamma. \quad (37)$$

We depict it with a diagram:

$$\begin{array}{ccc} M & \xrightarrow{X_{H_1, \dots, H_{n-1}}} & TM \\ \downarrow \pi & & \downarrow T\pi \\ N & \xrightarrow{X_{H_1, \dots, H_{n-1}}^\gamma} & TN \end{array}$$

(A curved arrow labeled γ points from N to M .)

Recall from former sections, the Hamiltonian vector field on a VNJ in adapted coordinates can be locally written as in (28). Then, the following theorem results:

Theorem 4. *The two vector fields $X_{H_1, \dots, H_{n-1}}^\gamma$ and $X_{H_1, \dots, H_{n-1}}$ are γ -related if and only if the following equation is satisfied:*

$$\begin{aligned} & \sum_{i=1}^{n-1} (-1)^{i-1} H_i \left(d(H_1 \circ \gamma) \wedge \dots \wedge d(\widehat{H_i \circ \gamma}) \wedge \dots \wedge d(H_{n-1} \circ \gamma) \right) \\ & + d(H_1 \circ \gamma) \wedge \dots \wedge d(H_{n-1} \circ \gamma) = 0. \end{aligned} \quad (38)$$

Proof. Referring to the coordinate expressions for a VNJ structure (25), we look for a section $\gamma = \gamma(x^1, \dots, x^{n-1}, \gamma^n(x^1, \dots, x^{n-1}))$ such that (37) is fulfilled. The projection of the vector field $X_{H_1, \dots, H_{n-1}}$ on M (28) onto N reads:

$$\begin{aligned}
X_{H_1, \dots, H_{n-1}}^\gamma &= \sum_{k=1}^{n-1} (-1)^{n-k} \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^k, \dots, x^n)} \frac{\partial}{\partial x^k} \\
&\quad + \sum_{i=1}^{n-1} (-1)^{i-1} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-2})} \frac{\partial}{\partial x^{n-1}} \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} (-1)^{n-j+i-2} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^j, \dots, x^{n-1})} \frac{\partial}{\partial x^j}
\end{aligned} \tag{39}$$

and its tangent image by the section γ is:

$$\begin{aligned}
T\gamma X_{H_1, \dots, H_{n-1}}^\gamma &= \sum_{k=1}^{n-1} (-1)^{n-k} \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^k, \dots, x^n)} \left(\frac{\partial}{\partial x^k} + \frac{\partial \gamma^n}{\partial x^k} \frac{\partial}{\partial x^n} \right) \\
&\quad + \sum_{i=1}^{n-1} (-1)^{i-1} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-2})} \left(\frac{\partial}{\partial x^{n-1}} + \frac{\partial \gamma^n}{\partial x^{n-1}} \frac{\partial}{\partial x^n} \right) \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} (-1)^{n-j+i-2} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^j, \dots, x^{n-1})} \left(\frac{\partial}{\partial x^j} + \frac{\partial \gamma^n}{\partial x^j} \frac{\partial}{\partial x^n} \right).
\end{aligned} \tag{40}$$

By direct comparison of (28) and (40), we obtain the following equation:

$$\begin{aligned}
&\sum_{k=1}^{n-1} (-1)^{n-k} \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^k, \dots, x^n)} \frac{\partial \gamma^n}{\partial x^k} - \frac{\partial(H_1, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-1})} \\
&\quad + \sum_{k=1}^{n-1} (-1)^{i-1} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, x^{n-2})} \frac{\partial \gamma^n}{\partial x^{n-1}} \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} (-1)^{n-j+i-2} H_i \frac{\partial(H_1, \dots, \hat{H}_i, \dots, H_{n-1})}{\partial(x^1, \dots, \hat{x}^j, \dots, x^{n-1})} \frac{\partial \gamma^n}{\partial x^j} = 0,
\end{aligned} \tag{41}$$

which can be rewritten in the following compact form

$$\begin{aligned}
&\sum_{i=1}^{n-1} (-1)^{i-1} H_i \left(d(H_1 \circ \gamma) \wedge \dots \wedge d(\widehat{H_i \circ \gamma}) \wedge \dots \wedge d(H_{n-1} \circ \gamma) \right) \\
&\quad + d(H_1 \circ \gamma) \wedge \dots \wedge d(H_{n-1} \circ \gamma) = 0,
\end{aligned} \tag{42}$$

which is precisely the equation given in (38). \square

Expression (38) receives the name of Hamilton–Jacobi equation on a volume Nambu–Jacobi manifold. We say that γ is a solution of the Hamilton–Jacobi problem on a volume Nambu–Jacobi manifold for (M, Ω, θ) .

Next, we consider a general Nambu–Jacobi manifold (M, Λ, \square) where the dimension of M is m and the order of the multivector Λ is n . Consider a

fibration $\pi : M \rightarrow N$ over an n -dimensional manifold N . Take $\gamma : N \rightarrow M$ a section of π such that $\pi \circ \gamma = \text{Id}_N$ and $\gamma(N)$ is a Lagrangian submanifold of (M, Λ, \square) . An additional hypothesis is that $\gamma(N)$ has a clean intersection with the leaves of the characteristic foliation \mathcal{C} defined by Λ . We recall that this implies that for each leaf $L \in \mathcal{C}$,

1. $\gamma(N) \cap L$ is a submanifold
2. $T(\gamma(N) \cap L) = T\gamma(N) \cap TN$

If we are assuming that $\gamma(N)$ is a j -Lagrangian submanifold of (M, Λ, \square) , then $\gamma(N) \cap L$ is a j -Lagrangian submanifold of L with the restricted Nambu–Jacobi structure, that is a volume structure. Consequently, $j = n - 1$ and N has dimension $n - 1$. Now, let H_1, \dots, H_{n-1} be Hamiltonian functions in M and $X_{H_1 \dots H_{n-1}}$ the corresponding Hamiltonian vector field.

We define the vector field on N ,

$$X_{H_1 \dots H_{n-1}}^\gamma = T\pi \circ X_{H_1 \dots H_{n-1}} \circ \gamma.$$

Since every Hamiltonian vector field is tangent to the characteristic foliation, one can conclude that

Theorem 5. *The vector fields $X_{H_1 \dots H_{n-1}}^\gamma$ and $X_{H_1 \dots H_{n-1}}$ are γ -related if and only if*

$$\begin{aligned} \sum_{i=1}^{n-1} (-1)^{i-1} H_i \left(d(H_1 \circ \gamma) \wedge \dots \wedge d(\widehat{H_i \circ \gamma}) \wedge \dots \wedge d(H_{n-1} \circ \gamma) \right) \\ + d(H_1 \circ \gamma) \wedge \dots \wedge d(H_{n-1} \circ \gamma) = 0. \end{aligned} \quad (43)$$

Therefore, (43) will be called the HJ equation for a general Nambu–Jacobi manifold and γ satisfying (43) will be a solution of the HJ problem on a general Nambu–Jacobi manifold for (M, Λ, \square) with a multivector Λ of order n and the dimension of the manifold M is m .

4 Application

Let us consider a four-dimensional system of first-order Riccati differential equations [7, 30, 32]. It is a system of four copies of the first-order Riccati differential equation on $\mathcal{O} = \{(x_1, x_2, x_3, x_4) \mid x_i \neq x_j, i \neq j = 1, \dots, 4\} \subset \mathbb{R}^4$, given by

$$\frac{dx^i}{dt} = a_0(t) + a_1(t)x^i + a_2(t)(x^i)^2, \quad i = 1, \dots, 4, \quad (44)$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary t -dependent functions. More generally, the term Riccati equation is used to refer to matrix equations with an analogous quadratic term, which occur in both continuous-time and discrete-time linear-quadratic-Gaussian control. The steady-state (non-dynamic) version of these is referred to as the algebraic Riccati equation [7, 30, 32].

These equations are compatible with several Hamiltonian structures, namely, the triples $(\mathcal{O}, \omega_l, H_l)$ such that H_l is a Hamiltonian function with respect to a presymplectic form ω_l .

For a VNJ structure, let us consider three of them for values $l = 2, 3, 4$ and the values of k range from 1 to 4.

It reads:

$$(\mathcal{O}, \omega_l, H_l) = \begin{cases} \mathcal{O} = \{(x_1, x_2, x_3, x_4) \mid x_i \neq x_j, i \neq j = 1, \dots, 4\} \subset \mathbb{R}^4, \\ \omega_l = \sum_{k < l} \frac{dx^k \wedge dx^l}{(x^k - x^l)^2} + \sum_{k > l} \frac{dx^l \wedge dx^k}{(x^l - x^k)^2} \\ H_l = \left(\sum_{k < l} \frac{x^k x^l}{x^k - x^l} + \sum_{k > l} \frac{x^l x^k}{x^l - x^k} \right) + b_1(t) \left(\sum_{k < l} \frac{1}{x^k - x^l} + \sum_{k > l} \frac{1}{x^l - x^k} \right) \end{cases}$$

To obtain a VNJ structure, we need to find a volume form Ω and a one-form θ compatible with $(\mathcal{O}, H_1, H_2, H_3)$. The procedure consists of retrieving (44) from the Nambu-Jacobi brackets

$$\dot{x}^i = \{H_1, H_2, H_3, x^i\}, \quad \forall i = 1, 2, 3, 4. \quad (45)$$

characterizing the evolution of the curves $(x^i(t)), i = 1, 2, 3, 4$. According to the theory of NJ structures, the computation of the bracket (45) according to (24) is

$$\dot{x}^i = \{H_1, H_2, H_3, x^i\} = (-1)^{4+i} \sum_{\sigma} (-1)^{\sigma(i_1, i_2, i_3)} \prod_{\sigma(i_j), k=1,2,3} \frac{\partial H_l}{\partial x^{\sigma(i_j)}} \quad (46)$$

for $i = 1, 2, 3, 4$, $j = 1, 2, 3$, $k = 1, 2, 3$ and $\sigma_{(i_1, i_2, i_3)}$ denotes all the permutations of elements $\{x^{i_1}, x^{i_2}, x^{i_3}\}$ when (i_1, i_2, i_3) take values from 1 to 4 and they are simultaneously different from i , namely, $i_1 \neq i_2 \neq i_3 \neq i$.

The argument takes the form:

$$\frac{\partial H_l}{\partial x^j} = F_{lj} \left(a_0(t) + a_1(t)x^l + a_2(t)(x^l)^2 \right),$$

$$\frac{\partial H_l}{\partial x^l} = F_{ll} \left(a_0(t) + a_1(t)x^k + a_2(t)(x^k)^2 \right)$$

for $k = 1, 2, 3, 4$ and $k \neq l$ and

$$F_{k, \sigma(i_j)} = \begin{cases} F_{lj} = \frac{1}{(x^l - x^j)^2}, & \text{when } \sigma(i_j) = j \neq l, \\ F_{ll} = \left(\sum_{k < l} \frac{1}{(x^k - x^l)^2} - \sum_{k > l} \frac{1}{(x^l - x^k)^2} \right), & \text{when } \sigma(i_j) = l. \end{cases}$$

By direct comparison between (44) and (46), we see there is difference of one factor, that implies that the canonical volume form $\Omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ for a four-dimensional system of first-order Riccati differential equations on a VNJ manifold is has to be conformally transformed to

$$\Omega = \begin{cases} F_{lj} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 & \text{when } \sigma(i_j) = j \neq l \\ F_{ll} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 & \text{when } \sigma(i_j) = l. \end{cases}$$

For this particular case, the canonical bracket $\{x^1, x^2, x^3, x^4\} = 1$ turns out in

$$\{x^1, x^2, x^3, x^4\} = \frac{1}{F_{lj}}, \quad \{x^1, x^2, x^3, x^4\} = \frac{1}{F_{ll}}$$

if $\sigma(i_j) = j \neq l$ and $\sigma(i_j) = l$, correspondingly. In this way, the canonical structure in (25) for (Λ, \square) when $n = 4$ has to be conformally transformed into $(\bar{\Lambda}, \bar{\square})$ that takes the expression

$$(\bar{\Lambda}, \bar{\square}) = \begin{cases} (F_{lj}\Lambda, F_{lj}\square), & \text{for } \sigma(i_j) = j \neq l \\ (F_{ll}\Lambda, F_{ll}\square), & \text{for } \sigma(i_j) = l. \end{cases}$$

This choice leaves the one-form $\theta = dx^4$ invariant. Hence, the VNJ structure for a four-dimensional system of first-order differential Riccati equations is $(\mathcal{O}, \bar{\Omega}, \theta)$, as defined above.

Let us now apply the HJ theory on this VNJ manifold. Recall that the definition of a Hamiltonian vector field on a NJ manifold is:

$$X_{H_1, \dots, H_{n-1}} = X_{H_1, \dots, H_{n-1}}^\Lambda + \sum_{i=1}^{n-1} (-1)^{i-1} H_i X_{H_1, \dots, H_{n-1}}^\square \quad (47)$$

Applied to our particular case of a four-dimensional system of first-order Riccati differential equations, this expression is simplified to:

$$X_{H_1 H_2 H_3} = \sharp_\Lambda (dH_1 \wedge dH_2 \wedge dH_3) + H_1 \square (dH_2 \wedge dH_3) - H_2 \square (dH_1 \wedge dH_3) + H_3 \square (dH_1 \wedge dH_2)$$

This can be rewritten in following compact form:

$$X_{H_1 H_2 H_3} = \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{12} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^3} + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{31} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^2} + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{23} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^1} + \left(\frac{\partial H_1}{\partial x^1} \frac{\partial H_2}{\partial x^2} \frac{\partial H_3}{\partial x^3} \right) \frac{\partial}{\partial x^4} \quad (48)$$

where $\Lambda_{mn} = \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^n}$ is an operator acting on the wedge product of two functions $H_j \wedge H_k = H_j \otimes H_k - H_k \otimes H_j$, each element of the scalar product in each entry of the operator. Notice that $j < k$, $j = 1, 2, 3$ and $i \neq j \neq k$ for each summand.

We define a projected vector field $X_{H_1 H_2 H_3}^\gamma$ as:

$$X_{H_1 H_2 H_3}^\gamma = T\pi \circ X_{H_1 H_2 H_3} \circ \gamma \quad (49)$$

such that the diagram below is commutative:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{X_{H_1 H_2 H_3}} & T\mathcal{O} \\ \downarrow \pi & & \downarrow T\pi \\ \mathcal{O}|_3 & \xrightarrow{X_{H_1 H_2 H_3}^\gamma} & T\mathcal{O}|_3 \end{array}$$

(A curved arrow labeled γ points from $\mathcal{O}|_3$ back up to \mathcal{O} .)

Let us choose a section γ that in coordinates takes the expression $\gamma = (x^1, x^2, x^3, \gamma^4(x^1, x^2, x^3))$ and we denote by \mathcal{O}_3 the restriction of the four-dimensional space $\{(x_1, x_2, x_3, x_4) | x_i \neq x_j, i \neq j = 1, \dots, 4\} \subset \mathbb{R}^4$ to $\{(x_1, x_2, x_3) | x_i \neq x_j, i \neq j = 1, \dots, 3\} \subset \mathbb{R}^4$. So, the projected vector field in coordinates reads

$$\begin{aligned} X_{H_1 H_2 H_3} &= \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{12} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^3} + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{31} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^2} \\ &\quad + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{23} \right)_l (H_j \wedge H_k)_l \frac{\partial}{\partial x^1} \end{aligned} \quad (50)$$

The image of this projected vector field by $T\gamma$ is:

$$\begin{aligned} T\gamma X_{H_1 H_2 H_3} &= \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{12} \right)_l (H_j \wedge H_k)_l \left(\frac{\partial}{\partial x^3} + \frac{\partial \gamma^4}{\partial x^3} \frac{\partial}{\partial x^4} \right) \\ &\quad + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{31} \right)_l (H_j \wedge H_k)_l \left(\frac{\partial}{\partial x^2} + \frac{\partial \gamma^4}{\partial x^2} \frac{\partial}{\partial x^4} \right) \\ &\quad + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{23} \right)_l (H_j \wedge H_k)_l \left(\frac{\partial}{\partial x^1} + \frac{\partial \gamma^4}{\partial x^1} \frac{\partial}{\partial x^4} \right) \end{aligned} \quad (51)$$

that compared with (48) provides the Hamilton–Jacobi equation for a four-dimensional system of first-order Riccati differential equations

$$\begin{aligned} &\sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{12} \right)_l (H_j \wedge H_k)_l \frac{\partial \gamma^4}{\partial x^3} + \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{31} \right)_l (H_j \wedge H_k)_l \frac{\partial \gamma^4}{\partial x^2} \\ &+ \sum_{l=1}^3 \left((-1)^{i+1} H_i \Lambda_{23} \right)_l (H_j \wedge H_k)_l \frac{\partial \gamma^4}{\partial x^1} = \frac{\partial H_1}{\partial x^1} \frac{\partial H_2}{\partial x^2} \frac{\partial H_3}{\partial x^3}. \end{aligned} \quad (52)$$

This equation is a quasi-linear equation that can be solved with the method of characteristics. Due to the volume of calculations, we just leave the equation indicated.

Acknowledgements

This work has been partially supported by MINECO MTM 2013-42-870-P and the ICMAT Severo Ochoa project SEV-2011-0087. We kindly acknowledge the committee for choosing our contribution to the Proceeding's book of the XXV International Workshop on Geometry and Physics (CSIC-Madrid, Spain). We thank Partha Guha for the suggestion of a Hamilton–Jacobi theory for Nambu–Jacobi manifolds.

References

- [1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, (2nd. ed. Benjamin-Cummings, Reading (Ma), 1978).
- [2] V.I. Arnold, *Mathematica methods of Classical Mechanics*, Graduate Texts in Mathematics **60**, (Springer–Verlag, Berlin, 1978).
- [3] F. Bayen and M. Flato, Remarks concerning Nambu's generalized mechanics, *Phys. Rev. D* **11** (1975), 3049–3053.
- [4] R. Camassa and D.D. Holm, An integrable shallow water equation with peaked solutions, *Phys. Rev. Lett.* **71** (1993), 1661–1664 .
- [5] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006), 1417–1458.
- [6] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010), 431–454.
- [7] J.F. Cariñena and J. de Lucas, Lie systems: theory, generalisations and applications, *Dissertationes mathematicae*, **479** (2011), 162pp.
- [8] B.A. Dubrovin and S.P. Novikov, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method, *Sov. Math. Dokl.* **27** (1983), 665–669.

- [9] D.B. Fairlie and I.A.B. Strachan, The Hamiltonian structure of dispersionless Toda hierarchy, *Physica D*, **90** (1996), 1–8.
- [10] V.T. Filippov, n -Lie algebras, *Sibirsk. Math. Zh.* **26** (1985), 126–140.
- [11] H. Goldstein, *Mecánica Clásica*, (4a Ed. Aguilar SA Madrid 1979).
- [12] J. Grawoski and G. Marmo, Remarks on Nambu–Poisson and Nambu–Jacobi brackets, *J. Phys. A: Math. Gen.* **32** (1999), 4239–4247
- [13] J. Grabowski and P. Urbanski, Tangent lifts of Poisson and related structures, *J. Phys. A* **28** (1995), 6743–6777.
- [14] P. Guha, Applications of Nambu mechanics to systems of hydrodynamic type II, *J. Nonlin. Math. Phys.* **11** (2004), 223–232.
- [15] R. Ibañez, M. de León, J.C. Marrero and D. Martín de Diego, Coisotropic and Legendre–Lagrangian submanifolds and conformal Jacobi morphisms, *J. Phys. A: Math. Gen.* **30** (1997), 5427–5444.
- [16] R. Ibañez, M. de León, J.C. Marrero and D. Martín de Diego, Dynamics of generalized Poisson and Nambu–Poisson brackets, *J. Math. Phys.* **38**, (1997), 2332–2344.
- [17] R. Ibañez, M. de León, J.C. Marrero and E. Padrón, Nambu–Jacobi and generalized Jacobi manifolds, *J. Phys. A: Math. Gen.* **31** (1998), 1267–1288.
- [18] M. de León, D. Iglesias-Ponte and D. Martín de Diego, Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems. *J. Phys. A: Math. Gen.* **1** (2008), 015205 14 pp.
- [19] M. de León, J.C. Marrero and D. Martín de Diego, *A geometric Hamilton–Jacobi theory for classical field theories*. In: *Variations, Geometry and Physics* 129–140, (Nova Sci. Publ., New York, 2009).
- [20] M. de León, J.C. Marrero and D. Martín de Diego, Linear almost Poisson structures and Hamilton–Jacobi equation: applications to nonholonomic Mechanics, *J. Geom. Mech.* **2**, (2010) 159–198.
- [21] M. de León, J.C. Marrero, D. Martín de Diego, M. Vaquero, A Hamilton–Jacobi theory for singular Lagrangian systems. *J. Phys. A* **54** (2013), 032902 32pp.

- [22] M. de León, D. Martín de Diego, J.C. Marrero, M. Salgado and S. Vilariño, Hamilton–Jacobi theory in k -symplectic field theories, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010), 14911507.
- [23] M. de León, D. Martín de Diego and M. Vaquero, A Hamilton–Jacobi theory for singular Lagrangian systems in the Skinner and Ruck setting. *Int. J. Geom. Meth. Mod. Phys.* **9** (2012), 125007 24pp.
- [24] M. de León, D. Martín de Diego and M. Vaquero, Hamilton–Jacobi in the Cauchy data space, *Reports on mathematical physics*, **76** (2015) 271-406.
- [25] M. de Leon, D. Martín de Diego and M. Vaquero, A Hamilton–Jacobi theory on Poisson manifolds, *J. Geom. Mech.* **6** (2014), 1450007 17pp.
- [26] M. de Leon and C. Sardón, A geometric Hamilton–Jacobi theory for a Nambu–Poisson structure, Accepted in *J. Math. Phys.* arXiv:1604.08904 (2016).
- [27] G. Marmo, G. Vilasi and A.M. Vinogradov, The local structure of n -Poisson and n -Jacobi manifolds, *Journal of Geometry and Physics* **25** (1998), 141-182.
- [28] Y. Nambu, Generalized Hamiltonian dynamics, *Phys. Rev. D* **7** (1973), 2405-2412.
- [29] P. Nevir and R. Blender, A Nambu representation of incompressible hydrodynamics using helicity and entropy, *J. Phys. A: Math. and Gen.* **26** (1993) 1189–1193.
- [30] J. Riccati, Animadversiones in aequationes differentiales secundi gradus, *Actorum Eruditorum quae Lipsiae publicantur*, Supplementa **8** (1724), 66–73.
- [31] D. Sahoo and M.C. Walsakumar, Nambu mechanics and quantization, *Phys. Rev. A* **46** (1992), 4410-4412
- [32] C. Sardón, Lie systems, Lie symmetry and reciprocal transformations, Manuscript accepted in *World Scientific Publ.*, arXiv:1508.00726 (2015).
- [33] L. Takhtajan, On foundation of the generalized Nambu Mechanics, *Comm. Math. Phys.* **160** (1994), 295–315.
- [34] W.M. Tulczyjew, Les sous-varietes Lagrangiennes et la dynamique Hamiltonienne, *C.R. Acad. Paris Ser. A* **283** (1976), 15–18.